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Annular Bounds for the Zeros of a Polynomial

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Abstract

In this paper we find annular bounds for the zeros of a polynomial .

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Introduction

A classic result on the zeros of polynomials is the following theorem due to Cauchy[5]:

Theorem A : Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1} + a_n z^n , a_n \neq 0$ be a complex polynomial. Then all the zeros of P(z) lie in the closed disk $|z| \leq 1 + M$, where

$$M = \max \left| \frac{a_j}{a_n} \right| , j = 0,1,2,......, n-1.$$

The bound for the zeros given by the above theorem was sharpened for a lacunary polynomial by Gulzar [4] , who proved the following result.

Theorem B: Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_p z^p + a_n z^n , a_p \neq 0, 1 \leq p \leq n-1$, be a polynomial of degree n. Then all the zeros of P(z) lie in $|z| \leq \max(1, k)$, where k $\neq 1$ is the positive root of the equation

$$z^{n+1} - z^n - Mz^{p+1} + M = 0$$

and $M = \max \left| \frac{a_j}{a_n} \right| , j = 0,1,2,......, p$.

In this paper we find annular bounds for the zeros of the polynomial given in Theorem B. In fact, we prove the following result:

Theorem 1: Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_p z^p + a_n z^n , a_p \neq 0, 1 \leq p \leq n-1$, be a polynomial of degree n. Then all the zeros of P(z) lie in $R_1 \leq |z| \leq R$,where

$$R_1 = \frac{-R^2 |a_1| (M' - |a_0|) + \sqrt{R^4 |a_1|^2 (M' - |a_0|)^2 + 4M'^3 R^2 |a_0|}}{2M'^2},$$

$$M' = \frac{|a_n|}{R-1} [R^{n+1} - R^n + MR^{P+1} - MR],$$

$$M = \max \left| \frac{a_j}{a_n} \right| , j = 0,1,2,......, p ,$$

R=1+k, and k$\neq 1$ is the positive root of the equation

$$z^{n+1} - z^n - Mz^{p+1} + M = 0 .$$

Taking p=n-1 in Theorem 1, we get the following result:

Corollary 1: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n$, be a polynomial of degree n. Then all the zeros of P(z) lie in $R_1 \le |z| \le R$,where

$$R_1=\frac{-R^2|a_1|(M'-|a_0|)+\sqrt{R^4|a_1|^2(M'-|a_0|)^2+4M'^3R^2|a_0|}}{2M'^2},$$

$$M'=\frac{|a_n|}{R-1}[R^{n+1}+(M-1)R^n-MR],$$

$$M=\max\left|\frac{a_j}{a_n}\right|, j=0,1,2,......,n-1,$$

R=1+k, and k$\neq 1$ is the positive root of the equation

$$z^{n+1}-(1+M)z^n+M=0 .$$

Taking $a_n=1$ in Cor. 1, we get the following result, which was partly proved by Chadia Affani-Aji et al [1] .

Corollary 2: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+z^n$ be a polynomial of degree n. Then all the zeros of P(z) lie in $R_1 \le |z| \le R$,where

$$R_1=\frac{-R^2|a_1|(M'-|a_0|)+\sqrt{R^4|a_1|^2(M'-|a_0|)^2+4M'^3R^2|a_0|}}{2M'^2},$$

$$M'=\frac{1}{R-1}[R^{n+1}+(M-1)R^n-MR],$$

$$M=\max|a_j|, j=0,1,2,......,n-1,$$

R=1+k, and k$\neq 1$ is the positive root of the equation

$$z^{n+1}-(1+M)z^n+M=0 .$$

Taking p=1 in Theorem 1, we get the following result:

Corollary 3: Let $P(z)=a_0+a_1z+a_nz^n$ be a polynomial of degree n. Then all the zeros of P(z) lie in $R_1 \le |z| \le R$,where

$$R_1=\frac{-R^2|a_1|(M'-|a_0|)+\sqrt{R^4|a_1|^2(M'-|a_0|)^2+4M'^3R^2|a_0|}}{2M'^2},$$

$$M'=|a_n|[R^n+MR],$$

$$M=\max\left|\frac{a_j}{a_n}\right|, j=0,1,$$

R=1+k, and k$\neq 1$ is the positive root of the equation

$$z^{n+1}-(1+M)z^n+M=0 .$$

Lemmas

For the proof of the above theorem , we need the following results:

Lemma 1: Let f(z) be analytic for $|z|\le 1$, f(0)=a , where $|a|<1$, $f'(0)=b, |f(z)|\le 1$ for $|z|=1$. Then, for $|z|\le 1$,

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$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)} .$$

The above lemma is due to Govil et al.[2].

Lemma 2: Let f(z) be analytic for $|z| \le R$, f(0)=0 , $f'(0) = b$ and $|f(z)| \le M$ for $|z| = 1$. Then, for $|z| \le R$,

$$|f(z)| \le \frac{M|z|}{R^2}\frac{M|z| + R^2|b|}{M + |z||b|}.$$

The above lemma is due to Govil and Jain [3, lemma 3], and follows by applying Lemma 1 to the function $\frac{f(Rz)}{M}$, which clearly satisfies the conditions of Lemma 1.

Proof of Theorem 1

In view of Theorem B, it suffices to show that P(z) does not vanish in $|z| < R_1$.

We have

$$P(z) = a_n z^n + a_p z^p + ... + a_1 z + a_0$$
$$= a_0 + g(z),$$

where

$$g(z) = a_n z^n + a_p z^p + ... + a_1 z .$$

Therefore , for $|z| = R > 1$,

$$|g(z)| = |a_n z^n + a_p z^p + ... + a_1 z|$$
$$\le |a_n||z|^n + |a_p||z|^p + ... + |a_1||z|$$
$$= |a_n|R^n + |a_p|R^p + ... + |a_1|R$$
$$= |a_n|[R^n + \left|\frac{a_p}{a_n}\right|R^p + ... + \left|\frac{a_1}{a_n}\right|R]$$
$$\le |a_n|[R^n + M\{R^p + ... + R\}]$$
$$= |a_n|[R^n + \frac{MR(R^p - 1)}{R-1}]$$
$$= \frac{|a_n|}{R-1}[R^{n+1} - R^n + MR^{P+1} - MR]$$
$$= M'.$$

Since g(z) is analytic for $|z| \le R$, g(0)=0, $g'(0) = a_1$ and $|g(z)| \le M'$ for $|z| = R$, it follows by lemma 2 that

$$|g(z)| \le \frac{M'|z|}{R^2}\frac{M'|z| + R^2|a_1|}{M' + |z||a_1|} = \frac{M'^2|z|^2 + M'R^2|z||a_1|}{R^2(M' + |z||a_1|)} \text{ for } |z| \le R .$$

Hence for $|z| \le R$,

$$|P(z)| = |a_0 + g(z)|$$

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$$\geq |a_0| - |g(z)|$$

$$\geq |a_0| - \frac{M'^2|z|^2 + M'R^2|z||a_1|}{R^2(M' + |z||a_1|)}$$

$$= -\frac{1}{R^2(M' + |z||a_1|)}[M'^2|z|^2 + R^2|z||a_1|(M' - |a_0|) - R^2M'|a_0|]$$

$$= -\frac{M'^2}{R^2(M' + |z||a_1|)}[|z| + \frac{R^2|a_1|(M' - |a_0|) + \sqrt{R^4|a_1|^2(M' - |a_0|)^2 + 4M'^3R^2|a_0|}}{2M'^2}]$$

$$\times[|z| + \frac{R^2|a_1|(M' - |a_0|) - \sqrt{R^4|a_1|^2(M' - |a_0|)^2 + 4M'^3R^2|a_0|}}{2M'^2}]$$

$$>0$$

if

$$|z| < \frac{-R^2|a_1|(M' - |a_0|) + \sqrt{R^4|a_1|^2(M' - |a_0|)^2 + 4M'^3R^2|a_0|}}{2M'^2} = R_1.$$

This shows that P(z) does not vanish in $|z| < R_1$ and the proof is complete.

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